Derivation of $E = mc^2$

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Over the years that Karl's Calculus Tutor has been on line, I have received a number of emails requesting that I show from first principles how you get Einstein's famous formula, $E = mc^2$. I shall begin by assuming that anybody interested enough to read this page is already familiar with basic Newtonian physics and is also familiar with the origin of the time dilation formula, the Lorenz contraction formula, and most importantly, the formula by which the mass of an object increases with its speed.

The time dilation formula, which states that a clock in motion ticks slower as seen by an observer at rest, derives from basic geometry combined with the experimentally verified fact that the speed of light in a vacuum is always the same regardless of whether the observer is moving toward or away from the light source or whether the light source is moving toward or away from the observer. The contraction of distances in the direction of motion and the increase in mass of an object that is in motion can both be thought of as consequences of the time dilation, since they are necessary to make the basic laws of physics remain consistent.

The mass, m, of an object whose mass at rest is m_0 and which is moving at a speed, v, with respect to the observer is given by

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Keep in mind that m_0 and the speed of light, c, are both constants.

In what follows, we shall use the above formula to show that when you increase the velocity of an object by a small amount, dv, sufficient to increase the mass by small amount, dm, then the kinetic energy is increased by an amount equal to $c^2 dm$.

Let's start by taking the derivative of the mass formula with respect to velocity. That gives

$$\frac{dm}{dv} = \frac{m_0 \frac{v}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

Observe the relationship between m and $\frac{dm}{dv}$ (which you can determine from simple algebra):

$$m = \frac{dm}{dv} \frac{\left(1 - \frac{v^2}{c^2}\right) c^2}{v} = \frac{dm}{dv} \frac{(c^2 - v^2)}{v}$$

So where do we get kinetic energy from? By classical physics, the kinetic energy is increased by the amount of work done on an object. Work is given as force applied to an object times the distance the object moves while the force is applied. This works also in the world of relativity. But before we can look at that, we have to be specific about what is meant by force.

You have probably already learned that Newton's second law is $f = ma$, or in words, that is force is mass times acceleration. In the Newtonian world, that works. But in the world of relativity it doesn't. Why? Because it assumes that mass is constant. In the Newtonian world, the mass of objects does not change as a function of their speed. But in relativity it does. So in relativity we use a definition of force that works both in relativity and in the Newtonian world. It is that force is the rate of change of momentum.

$$f = \frac{dp}{dt}$$

where momentum, p, is given by $p = mv$. If you assume mass to be constant and take the derivative of momentum, you easily see that you get the familiar $f = ma$, where acceleration, a, is the time derivative of velocity, v. But if mass is not constant, but is instead a function of velocity, then you get (applying the product rule):

$$f = \frac{dp}{dt} = \frac{d\,(mv)}{dt} = v\frac{dm}{dt} + m\frac{dv}{dt}$$

Now we replace m with the expression we derived earlier for it in terms of $\frac{dm}{dv}$:

$$f = \frac{d\,(mv)}{dt} = v\frac{dm}{dt} + \frac{dm}{dv}\,\frac{(c^2 - v^2)}{v}\,\frac{dv}{dt}$$

By the chain rule we also have

$$\frac{dm}{dt} = \frac{dm}{dv}\,\frac{dv}{dt}$$

which gives

$$f = \frac{d\,(mv)}{dt} = v\frac{dm}{dt} + \frac{(c^2 - v^2)}{v}\,\frac{dm}{dt}$$

Now we drop the middle part of the equation and just keep the right and left parts. And we multiply through by dt, which gives what happens when a force is applied to the object over a short time, dt:

$$f\,dt = v\,dm + \frac{(c^2 - v^2)}{v}\,dm$$

But this is not what we are looking for. We want the left side to be force applied over a short distance, dx. To convert dt to dx, we multiply by velocity, $\frac{dx}{dt}$. On the left that gives us $f\,dx$, which is the change in kinetic energy, dE (recall that work is force applied over distance, and change in kinetic energy is equal to work done). We multiply the right by v, which is equal to $\frac{dx}{dt}$.

$$dE = f\,dx = v^2\,dm + \left(c^2 - v^2\right)\,dm$$

The v^2's cancel, leaving

$$dE = c^2\,dm$$

Remember that c, and therefore also c^2, is constant. So this indicates that the amount of kinetic energy it takes to cause an object to gain any amount of mass is equal to the amount of mass gained times c^2, which you get by integrating the last equation above.