8.2 The Dance of the Derivatives

How to Solve Related Rates Problems

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A Shaggy Dog Story

That little doggie looked so cute when you first brought him home. How could any creature so adorable make such a mess of your house? Here is his latest (though by far not his messiest) caper. It seems he has discovered the bathroom. And in it is this wonderful toy hanging on the wall with a soft end dangling down that is just begging to be pulled. Which is what he is now in the habit of doing. He trots into the bathroom, grabs the end of the toilet paper, and then runs out the door with a streamer unreeling behind him.

With all this happening, amazingly you are thinking about calculus and asking yourself, "If the dog runs at 2 meters per second, how quickly is the radius of the toilet paper roll decreasing?" Of course you are.

Before we dive into the how to of this problem, we need a little more information. One is how thick is the toilet paper. Why do we need to know this? Well if the paper were very thick then we'd expect that each turn of the roll would reduce the radius by more than if the paper were thin. We'll say that this particular brand is 0.00005 meters thick (that's about 2 mils in the English system of measures). Also since the radius of the roll is changing at a rate which is perhaps not constant, we need the problem to state at exactly what moment we want to know this rate of change. So let's say it is at the moment when the radius of the roll is 0.05 meters (about 2 inches).

The first trick in this problem is to relate the radius of the roll to how much toilet paper the dog has already pulled off. Visualize the roll in crossection. It has a crossectional area of $A = \pi r^2$, where A is area and r is radius. I could, at this point, substitute 0.05 meters for r, but usually it is good practice to wait until the end of the problem before you substitute values for variables. So that's what we'll do.

Now consider the thickness of the toilet paper, ξ (which the problem stated was equal to 0.00005 meters), and think about how much area a length, s, of toilet paper of that thickness contributes to the crossectional area of the roll. If you look at that piece of toilet paper edge on, it is a very long and thin rectangle. And the area of that rectangle is thickness times length, or ξs.

The rate at which the dog pulls toilet paper off the roll is given in the problem. And that rate is the time derivative (where t is the time variable) of the length of paper he has already pulled.

$$\frac{ds}{dt} = 2\,\frac{\text{meters}}{\text{second}} \tag{8.2-1}$$

Do you see what that equation is saying? The rate at which length of paper, s, is being pulled off the role with respect to time, t, is 2 meters per second.

With each meter of paper the dog pulls off, the crossectional area of the roll is reduced by ξ times one meter. The rate at which he pulls the paper off is given in eq. 8.2-1 in meters per second. So the equation for how fast the crossectional area of the roll is reduced is

$$\frac{dA}{dt} = -\xi\,\frac{ds}{dt} \tag{8.2-2}$$

What this equation says is that the crossectional area of the roll, A, is *reduced* (that's where the minus sign comes from), and the *rate with respect to time*, t, that it is reduced is the paper's thickness, ξ, times the rate it is being pulled off, $\frac{ds}{dt}$, in meters per second. Note that $\frac{dA}{dt}$ has units of square meters per second.

But how is all this crossectional area stuff related to the radius of the roll? The crossection of the roll is a circle. So we get this relationship from how the area of a circle is related to its radius.

$$A = \pi r^2 \tag{8.2.3-a}$$

But both A and r are functions of time, t. That is, it would be reasonable to express them as $A\left(t\right)$ and $r\left(t\right)$ respecively. So another more explicit way to write this is

$$A\left(t\right) = \pi r^2\left(t\right) \tag{8.2.3-b}$$

Now look at what happens when you take the time derivative of both sides of equation 8.2-3b. Clearly on the left we get $\frac{dA}{dt}$. On the right, though, we have a function of a function – that is a composite. It is a composite of the radius function of time, $r(t)$, and the function of squaring the radius. So we have to apply the chain rule.

The derivative of squaring the radius is $2r(t)$. And the derivative of the radius function itself, $r(t)$, is $\frac{dr}{dt}$. So the derivative of the radius function squared is $2r(t)\frac{dr}{dt}$. That gives us, for the entire equation of the derivative of equation8.2-3b:

$$\frac{dA}{dt} = 2\pi r(t)\frac{dr}{dt} \tag{8.2-4}$$

Observe what the problem asks for. It is asking for the rate at which the radius of the roll decreases. Isn't that $\frac{dr}{dt}$? So we need to solve for that:

$$\frac{dA}{dt}\frac{1}{2\pi r(t)} = \frac{dr}{dt} \tag{8.2-5a}$$

We already have an expression for $\frac{dA}{dt}$ from equation 8.2-2. So we substitute that in for $\frac{dA}{dt}$.

$$-\xi\frac{ds}{dt}\frac{1}{2\pi r(t)} = \frac{dr}{dt} \tag{8.2-5b}$$

And now it all works out because $\frac{ds}{dt}$, ξ, and $r(t)$ are given in the problem. Remember

$$\begin{aligned}\frac{ds}{dt} &= 2\,\frac{\text{meters}}{\text{second}}\\ f(t) &= 0.05\,\text{meters}\\ \xi &= 0.00005\,\text{meters}\end{aligned}$$

So finishing the solution is just a matter of plugging in values. My calculator tells me that

$$\frac{dr}{dt} = -0.0003183\,\frac{\text{meters}}{\text{second}} \tag{8.2-5c}$$

which is about -0.012 inches per second. Observe that the result is negative. This indicates that the radius of the roll is decreasing with time, as we would expect.

Seeing that the sign of the result is as you would expect it is just one of the things you can do to check your solution for correctness. Here are some others:

1. If you increase the thickness, ξ, of the paper, do you expect that the roll will shrink faster or slower? It would shrink faster because each turn of the roll takes off more area from the crossection of the roll. Now look at equation 8.2-5b. Does it predict that increasing the thickness makes the roll shrink faster?

2. If the dog runs faster, that is if you increase $\frac{ds}{dt}$, do you expect the roll to shrink faster or slower? Faster of course. Does equation 8.2-5b predict that?

3. If the radius, $r(t)$, is smaller, do we expect the roll to shrink faster or slower? Think about this one. With a smaller radius it's circumference is less, so with each meter of paper pulled you will get more turns of the roll. So the roll shrinks faster when it is smaller. Does the way $r(t)$ appears in equation 8.2-5b predict this?

Perhaps you have not been in the habit of doing checks like the ones I have shown above. Now is a good time to start. You will save yourself a lot of wrong answers by performing reasonableness checks on all your solutions to problems, especially if you are headed into engineering or the sciences. Remember that if someday you are solving problems as part of your profession, there will be no answers in the back of the book for you to check against. You will have to know whether or not your answer is right on your own.

The Seven Veils: The Questions to Ask Yourself When Doing Related Rates Problems

Question 1: What is the independent variable in this problem? The independent variable is usually (but not always) time, which you traditionally notate using the symbol, t. The key phrases that indicate time to be the independent variable are *"how fast," "how often," "at what speed," "at what rate is this or that changing,"* and so on. In rare cases a related rate problem might have something other than time as its independent variable. If, for example, the problem asks "how much water boils for each gallon of fuel," you can be relatively sure that the independent variable is gallons of fuel. The word, *"per,"* or the phrase, *"for each,"* gives you a clue that the word or phrase that follows is likely to be an independent variable.

Question 2: What dependent variables does the problem reveal? In the case of the doggie problem above, radius of the roll is clearly a dependent variable, since the problem asks you about it. Anything that the problem mentions that is a function of the independent variable you arrived at by asking yourself the first question is probably going to be a dependent variable. Since the problem mentions the length of paper being pulled (or at least the rate of length of paper being pulled) that means that length of paper being pulled is also a dependent variable.

Question 3: What other dependent variables (if any) does the problem imply? For this you have to use your own knowledge of how the world works. Often this is not an easy question to answer. In the case of the doggie problem, the implied dependent variable was the crossectional area of the roll. We were only able to arrive at this by thinking carefully about what happens when you pull paper off a roll. In other problems you will have to use your knowledge of geometry or trig to answer this question. But answering it always starts with taking a moment to clearly visualize what is going on in the problem. And quite often, the problem will indeed state all the variables you need in order to solve it, in which case the answer to this question will be none.

Question 4: What are the relationships between the variables? Again this often involves tapping your world knowledge as well as what you learned in geometry and trig. In the doggie problem, we have real-world knowledge that toilet paper rolls usually have a circular crossection. And we know from junior high school that the area of a circle is πr^2. That gives us the relationship between dependent variable, radius, and dependent variable, area.

Question 5: What do you get when you take the derivative with respect to the independent variable of both sides of the relationship equation(s)? This just means you have to take some derivatives. You will almost always have to apply the chain rule, because one of your dependent variables will very likely be squared or cubed or square rooted, or have some trig function taken of it. Here are some examples using t as the independent variable:

1. If you have dependent variable, y, and you see the expression, $y^3 - y$, then the derivative of that expression is:

$$\left(3y^2 - 1\right) \frac{dy}{dt}$$

2. If you have dependent variable, θ, and you see the expression, $\sin(\theta)$, then the derivative of that expression is:

$$\cos(\theta) \frac{d\theta}{dt}$$

3. If you have dependent variables, u and v, and you see the expression, uv, you have to apply the product rule, and you will get as the derivative of uv:

$$u\,\frac{dv}{dt} + v\,\frac{du}{dt}$$

Each of the above is an example of implicit differentiation.

Question 6: What is the problem asking for? Very often it is asking for a rate. That means it is asking for the derivative of one of your dependent variables. In the doggie problem, it asked for the rate at which the radius decreases. We see “rate” and “radius” in the same phrase, and that indicates that the problem is asking for the derivative of radius with respect to the independent variable, which in this case is time. We declared r to be our symbol for the dependent variable, radius, and t to be our symbol for the independent variable, time. So what the problem is asking you to solve for is $\dfrac{dr}{dt}$. Look at the equations you have and identify where this term occurs and how you might solve for it in terms of the other variables. Then do so.

Question 7: What variables and derivatives does the problem give actual values for? In the doggie problem it stated the thickness of the paper, the rate the paper was being pulled, and the radius at which you were to establish the rate (observe that the thickness is neither a dependent nor an independent variable – it is a constant. That is because it does not change as the dog pulls the paper.) We did identify length of paper pulled as a dependent variable. We observe that the problem gives a rate for that, which means that it assigns a value to the *derivative* of length of paper pulled. It also assigns a value to the radius (and not its derivative). Once you have identified all the values given, go back to the equation where you solved for what the problem was asking for and plug all the values in. That will enable you to get an answer.

Some Worked Examples

Example 1: Two taxicabs begin at the same time at the intersection of Park Ave and 59^{th} St. One taxicab heads uptown (North) on Park Ave at 40 mph. The other heads west on 59^{th} St at 30 mph. Both have the unbelievable good fortune not to be stopped by traffic lights or slowed by traffic. When the taxicab on Park Ave. has gone half a mile, how fast is the distance (as the New York crow flies) between them increasing?

Answering Question 1: The problem asked *“how fast.”* So the independent variable is time, t.

Answering Question 2: The problem talks of distance, and indeed we have distance north of Park and 59^{th} and we have distance west of Park and 59^{th}. We'll call them y and x respectively. The problem asks for the distance between the taxicabs, so that is also a dependent variable, which we shall call, s. Observe that these are all functions of time, since both taxicabs are moving.

Answering Question 3: There are no hidden aspects of taxicabs that are relevant to this problem, so there are no inferred variables.

Answering Question 4 There are two relationships you need to know about in this problem. One is that the taxicabs are traveling at right angles to each other. This means that the Pythagorean distance formula applies. The distance, s, between the taxicabs is the hypotenuse of the right triangle formed by the line connecting the northbound taxicab with Park and 59^{th} and the line connecting the westbound taxicab with Park and 59^{th}. The lengths of those sides are y and x respectively. So you have the relationship:

$$s^2 = x^2 + y^2$$

There is another relationship in this problem that we can infer from our experience doing motion problems in junior high school. The problem gives us the speeds of the taxicabs and that they start from the same point at the same time. Remember that the northbound taxicab goes 40 mph and the westbound one goes 30 mph. So we can conclude that at any time:

$$\frac{y}{40} = \frac{x}{30} \tag{8.2-6}$$

or indeed that

$$\frac{3}{4}\frac{dy}{dt} = \frac{dx}{dt} \tag{8.2-7}$$

which is the same as saying that three fourths of the speed of the northbound taxicab is equal to the speed of the same as the speed of the westbound taxicab. And we already knew that.

Answering Question 5: This just means taking the derivatives of the relationship equations we got when we answered question 4. For the Pythagorean relationship (eq. 8.2-5), that gives:

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \tag{8.2-8}$$

Taking the derivative of the other relationship yields

$$\frac{3}{4}\frac{dy}{dt} = \frac{dx}{dt} \tag{8.2-9}$$

Answering Question 6: The problem is asking for the rate at which the distance, s, between the two taxicabs is increasing. So it is asking for $\frac{ds}{dt}$. Observe that this parameter already appears in one of our equations. To solve for it, we simply have to divide the derivative of the Pythagorean equation (eq. 8.2-8) by $2s$.

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) \tag{8.2-10}$$

Answering Question 7: Now all we have to do is plug in numbers for all the parameters other than $\frac{ds}{dt}$ into the above and we will have a value for $\frac{ds}{dt}$. So what numbers were given in the problems? Only that northbound speed is 40 mph:

$$\frac{dy}{dt} = 40$$

and the westbound speed is 30 mph:

$$\frac{dx}{dt} = 30$$

Also that at the moment we want to know the speed, the northbound taxicab is half a mile north of Park and 59^{th}:

$$y = 0.5$$

In order to complete this step we still have to know x and s. We already determined that $x = \frac{3}{4}y$. Substituting 0.5 for y we find $x = 0.375$. As for s, we have the Pythagorean distance formula (eq. 8.2-5) to determine s from x and y:

$$s = \sqrt{x^2 + y^2} = \sqrt{0.140625 + 0.25} = 0.625\,\text{miles} \tag{8.2-11}$$

When I put all the numbers into the equation that resulted from answering question 6, I get $\frac{ds}{dt} = 50\,\text{mph}$

Example 2: A solution drains through a filter-funnel at a rate of 10 $\frac{\text{cc}}{\text{minute}}$. From the center point of its mouth to its apex, the funnel is 10 cm. At its mouth it is 12 cm in diameter. How fast is the solution-level dropping when there are 200 cc left in the funnel? (Assume that the funnel is a right circular cone) For reference, the volume of a cone is given by

$$V = \frac{1}{3}\pi r^2 h \tag{8.2-12}$$

where V is the volume, r is the radius of the base, and h is the cone's height from base to apex.

Answering Question 1: The problem asks, *"how fast,"* so clearly the independent variable is time, t.

Answering Question 2: The problem mentions the solution-level, the volume, and the rate at which volume is being lost. So clearly solution-level, h, and volume, V, are dependent variables. The rate at which volume is being lost is the derivative of volume, so it is $\frac{dV}{dt}$.

Answering Question 3: Think about the situation. You know that the funnel is a cone. But the volume of solution in the funnel is also a cone, although this cone shrinks as the solution flows out. The top surface of the solution is a circle. And the amount of volume of solution is related to the radius of that circle by the volume formula for a cone (given above). So that radius, r, is also a dependent variable.

Answering Question 4: The volume of a cone formula is a relationship that the problem gave us. Again it is:

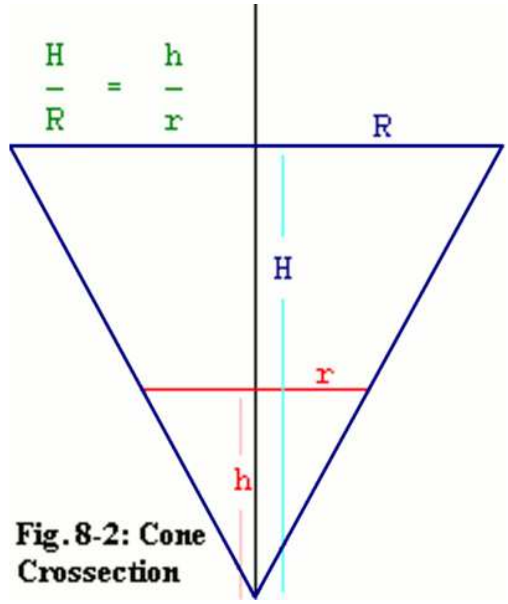


Fig. 8-2: Cone Crossection

There is another relationship you can get by what you know about cones. Since the diameter of the base of the funnel is given as 12 cm, clearly the radius of the base is 6 cm. The height is given as 10 cm. So the ratio of height to radius is $\frac{10}{6} = \frac{5}{3}$. But it is an intrinsic property of cones that *the radius of its skirt increases in linear proportion to the distance along the axis from its apex*. That means that if the ratio of the big funnel-cone's height to base-radius is $\frac{5}{3}$, then the smaller cone that is the volume of solution must have the *same ratio* of height to base-radius. Figure 8-2 shows how the triangular crossection of a cone leads to this conclusion (recall studying similar triangles in geometry). The height of the smaller cone is the solution level, which we have called, h, and is a dependent variable in this problem. The base radius of that smaller cone we have called, r, and it is also a dependent variable. From this discussion we now know that:

$$r = \frac{5}{3}\,h \tag{8.2-13}$$

Answering Question 5: There are two ways you could go at this point. One is simply to take the derivative of the volume-of-a-cone formula (eq. 8.2-12), which is the hard way, but does lead to a solution. I'll show you the easy way first, and then demonstrate that the hard way gets you to the same place. The easy way is to observe that you can make a substitution for r before you take the derivative of the volume formula:

$$V = \frac{1}{3}\,\pi\left(\frac{5}{3}\,h\right)^2 h = \frac{3}{25}\,\pi h^3 \tag{8.2-14}$$

Now that we have reduced the number of variables by one we take the derivative of the above:

$$\frac{dV}{dt} = \frac{9}{25}\,\pi h^2\,\frac{dh}{dt} \tag{8.2-15}$$

But suppose we had just taken the derivative of volume formula without first making the convenient substitution. Since the volume formula (eq. 8.2-12) contains the product of two dependent variables (that is it contains r^2h) we have to use the product rule. When you do you get

$$\frac{dV}{dt} = \frac{1}{3}\,\pi\left(2rh\,\frac{dr}{dt} + r^2\frac{dh}{dt}\right) \tag{8.2-16}$$

If you made the substitution at this point, that is replaced r with $\frac{3}{5}h$ into eq. 8.2-16, what would happen? You'd still be left with a $\dfrac{dr}{dt}$ term, but you can substitute that as well by taking the derivative of:

$$r = \frac{3}{5}\,h \tag{8.2-17a}$$

to find that

$$\frac{dr}{dt} = \frac{3}{5}\,\frac{dh}{dt} \tag{8.2-17b}$$

I'll let you take that substitution the rest of the way. If you make no mistakes, you will end up with the exact same equation we got by substituting first and then taking the derivative.

Answering Question 6: The problem asks for the rate at which the solution level is dropping. The solution level is given by the dependent variable, h. So the rate of change of the solution level is $\dfrac{dh}{dt}$. And so that is what we solve for.

$$\frac{dh}{dt} = \frac{dV}{dt}\,\frac{25}{9\pi}\,\frac{1}{h^2} \tag{8.2-18}$$

Answering Question 7: The problem gives $\dfrac{dV}{dt} = -10\,\frac{\text{cc}}{\text{minute}}$ (the minus sign tells you that the volume in the funnel is decreasing). But the problem doesn't give a value for h. So what do we do? The problem does tell us to find the rate that the level is decreasing *when the volume of solution left in the funnel is 200 cc.* That is the same as saying, when $V = 200\,\text{cc}$. And we know from eq. 8.2-14 that

$$V = \frac{3}{25}\,\pi h^3 \tag{8.2-19}$$

From that you can solve for h by:

$$h = \left(\frac{25}{3\pi}\,V\right)^{\frac{1}{3}} \tag{8.2-20}$$

Since V is given, from here it's just a matter of poking your calculator keys to get an answer. I get $h = 8.0953\,\text{cm}$, and $\dfrac{dh}{dt} = -0.13492\,\frac{\text{cm}}{\text{minute}}$ (note that the minus sign indicates that the level is going down, as we would expect if the solution is draining out).

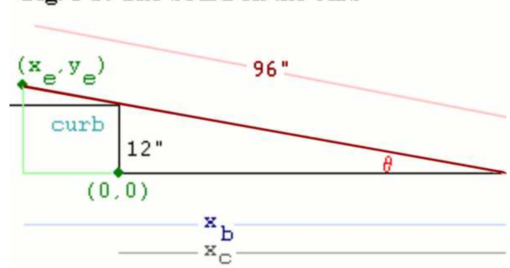
Example 3: A board that is 96 inches long rests with one end on top of a 12 inch high curb. The other end rests on the pavement beneath the curb, which is assumed to be flat. The lower end is slid along the pavement toward the curb at a rate of 1 inch per second. When there is 36 inches left between the end of the board and the base of the curb, how fast is the free end of the board moving? And at what rate is the angle of the board changing?

This problem is a step up in level of difficulty from the two that preceded it. But the difference is tantamount to the difference between untying one knot and untangling an entire snarled wad of yarn. The techniques are the same, you just have to do more of them in this problem than in the last two. But like the snarled wad of yarn, you find an end then methodically work it back through all the loops and tangles, and if you keep at it long enough it will be unsnarled.

Answering Question 1: Again the problem is asking *"how fast"* and *"at what rate,"* so it is clear that the independent variable is time, t.

Answering Question 2: The position of the lower end of the board, or more specifically its distance from the curb, is clearly a dependent variable. Let's call that x_c. The rate at which that end moves is also the rate at which x_c decreases. And that rate is $\dfrac{dx_c}{dt}$. There is also the position of the other end of the board. But it has both a horizontal and vertical position. So let's call them x_e and y_e respectively, and we shall reference them from the base of the curb. Finally there is the angle of the board, θ, which we shall consider in radians referenced from horizontal. Figure 8-3 shows all the dimensions of this problem. Observe that the way the diagram is drawn, x_e is *negative.*

Fig. 8-3: The board on the curb



Answering Question 3: When you draw a diagram of this problem, it becomes clear that there is an important variable that is not mentioned in the problem. That is the horizontal distance from the lower end of the board to the spot on the ground immediately below the upper end of the board. Call this x_b. Also, if the angle of the board is a dependent variable, perhaps we might consider the slope of the board also as a dependent variable, m.

Answering Question 4: So what are the relationships among the variables here? Clearly there is a right triangle present here, so the Pythagorean distance formula applies. From the diagram we can see that

$$96^2 = x_b^2 + y_e^2 \tag{8.2-21}$$

Now look at the two base lengths, x_b and x_c. Notice that the length from $(0, 0)$ to $(x_c, 0)$ added to the length from $(0, 0)$ to $(x_e, 0)$ is exactly equal to the length, x_b. But since x_e is negative, the relationship is:

$$x_c - x_e = x_b \tag{8.2-22}$$

Look at the diagram again. Do you see the similar triangles? The length, x_c, forms the smaller triangle, and the length, x_b, forms the larger one. By the rule for similar triangles we have:

$$\frac{y_e}{x_b} = \frac{y_e}{x_c - x_e} = \frac{12}{x_c} = -m \tag{8.2-23}$$

Observe that we put a minus sign in front of the slope, m, because the way the board is drawn in the diagram, it is sloping down. And finally, from trigonometry we have

$$-m = \tan(\theta) \qquad \text{or equivalently} \qquad \arctan(-m) = \theta \tag{8.2-24a}$$

Again observe that there is a minus sign on the slope, m, because the board is sloping *down* the way it is drawn in the diagram. You also have by trigonometry:

$$96\,\cos(\theta) = x_b \tag{8.2-24b}$$

Answering Question 5: All we need to do for this one is take the derivatives of each of the relationships listed in the question 4 paragraph above. So taking the derivative of both sides of eq. 8.2-21:

$$0 = 2x_b\,\frac{dx_b}{dt} + 2y_e\,\frac{dy_e}{dt} \tag{8.2-25a}$$

from which you can divide out a factor of 2 to get:

$$0 = x_b\,\frac{dx_b}{dt} + y_e\,\frac{dy_e}{dt} \tag{8.2-25b}$$

Taking the derivative of eq. 8.2-22 is pretty easy:

$$\frac{dx_c}{dt} - \frac{dx_e}{dt} = \frac{dx_b}{dt} \tag{8.2-26}$$

On eq. 8.2-23, it turns out the we will mostly be interested in taking the derivative of $\frac{12}{x_c} = -m$:

$$-\frac{12}{x_c^2}\frac{dx_c}{dt} = -\frac{dm}{dt} \tag{8.2-27a}$$

and the minus signs cancel to give:

$$\frac{12}{x_c^2}\frac{dx_c}{dt} = \frac{dm}{dt} \tag{8.2-27b}$$

From the trig equations, the arctan equation in (8.2-24a) turns out to be more useful than the tan equation. Its derivative (remembering that the derivative of $\arctan(x)$ is $\frac{1}{1+x^2}$) is

$$\frac{-1}{1+m^2}\frac{dm}{dt} = \frac{d\theta}{dt} \tag{8.2-28a}$$

We also have from 8.2-24c and 8.2-24d respectively:

$$-96\sin(\theta)\frac{d\theta}{dt} = \frac{dx_b}{dt} \tag{8.2-28b}$$

$$96\cos(\theta)\frac{d\theta}{dt} = \frac{dy_e}{dt} \tag{8.2-28c}$$

Answering Question 6: Although this problem is harder than the other two, it is not the calculus part that makes it harder. We have already done all the calculus we need to do on this problem. All that is left to be done is the algrebra and trig. There is more of it to do on this problem than on the others, but it is still stuff you have had before. The problem asked for the speed of the free end, which will be a combination of $\frac{dx_e}{dt}$ and $\frac{dy_e}{dt}$. And it also asked for the rate at which the angle, θ, is changing, which would be $\frac{d\theta}{dt}$. At this point we have all the equations we need to solve all this. All we need to do is organize them in a way that makes sense and substitute one into another until we get a solution.

It turns out that $\frac{d\theta}{dt}$ is easier to solve for than the either of the other two variables. So let's do that one first. Remember that the distance from the low end of the board to the base of the curb, x_c, is given. So is the rate at which it is being pushed, $\frac{dx_c}{dt}$. From those two we can get the slope of the board, m, from eq. 8.2-23 and we can get the rate at which that slope is changing, $\frac{dm}{dt}$, from eq. 8.2-27b:

$$\frac{y_e}{x_b} = \frac{y_e}{x_c - x_e} = \frac{12}{x_c} = -m \tag{8.2-23}$$

$$-\frac{12}{x_c^2}\frac{dx_c}{dt} = -\frac{dm}{dt} \tag{8.2-27b}$$

Once we know m and $\frac{dm}{dt}$, eq. 8.2-28b will give us $\frac{d\theta}{dt}$:

$$\frac{-1}{1+m^2}\frac{dm}{dt} = \frac{d\theta}{dt} \tag{8.2-28a}$$

We have expressions for $\frac{dx_b}{dt}$ and $\frac{dy_e}{dt}$ in equations 8.2-28c and 8.2-28d. But they involve the sine and cosine of θ, which we do not yet know. But we do know what $\tan(\theta)$ is from eq. 8.2-24a. And, remembering that $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\sin^2(x) + \cos^2(x) = 1$, you can easily derive:

$$\cos(\theta) = \frac{1}{\sqrt{1+\tan^2(\theta)}} \tag{8.2-29a}$$

$$\sin(\theta) = \frac{\tan(\theta)}{\sqrt{1+\tan^2(\theta)}} \tag{8.2-29b}$$

And don't just take my word for it. Derive these for yourself. Eq. 8.2-24a tells us that $-m = \tan(\theta)$. So from the trig identities in equations 8.2-29a and 8.2-29b we have:

$$\cos(\theta) = \frac{1}{\sqrt{1+m^2}} \tag{8.2-30a}$$

$$\sin(\theta) = \frac{-m}{\sqrt{1+m^2}} \tag{8.2-30b}$$

Substituting those into equations 8.2-28c and 8.2-28d and substituting $\frac{d\theta}{dt}$ from eq. 8.2-28b, you have:

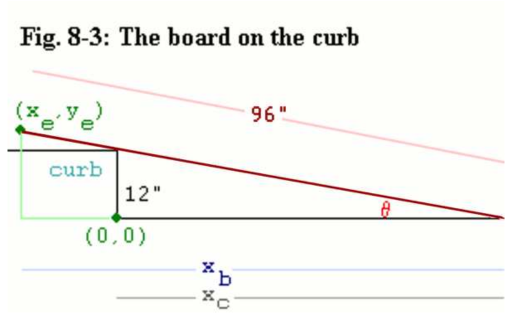
$$96\,\frac{-m}{(1+m^2)^{\frac{3}{2}}}\frac{dm}{dt} = \frac{dx_b}{dt} \tag{8.2-30c}$$

$$96\,\frac{-1}{(1+m^2)^{\frac{3}{2}}}\frac{dm}{dt} = \frac{dy_e}{dt} \tag{8.2-30d}$$

Finally we observe that we can get $\dfrac{dx_e}{dt}$ using eq. 8.2-26. Solving that for $\dfrac{dx_e}{dt}$ gives:

$$\frac{dx_c}{dt} - \frac{dx_b}{dt} = \frac{dx_e}{dt} \tag{8.2-31}$$

Answering Question 7: The problem tells you that $x_c = 36$ inches. It also tells you that $\dfrac{dx_c}{dt} = -1\,\frac{\text{inch}}{\text{sec}}$ (it's negative because the low end of the board is moving to the left so x_c is decreasing). Now we take it step by step. First, eq. 8.2-23 tells us that $m = -\frac{1}{3}$. From eq. 8.2-27b you get $\dfrac{dm}{dt} = -\frac{1}{108}$. From eq. 8.2-28b you get $\dfrac{d\theta}{dt} = \frac{1}{120}\,\frac{\text{radian}}{\text{sec}}$. From eq. 8.2-30c you get $\dfrac{dx_b}{dt} = -0.2529822\,\frac{\text{inches}}{\text{sec}}$ (that is, x_b is decreasing), and from eq. 8.2-30d you get $\dfrac{dy_e}{dt} = 0.7589466\,\frac{\text{inches}}{\text{sec}}$ (that is the free end of the board is getting higher). From eq. 8.2-31 you have $\dfrac{dx_e}{dt} = -0.7470178\,\frac{\text{inches}}{\text{sec}}$ (that is the high end of the board is moving to the left).



Finally, now that we have values for $\dfrac{dx_e}{dt}$ and $\dfrac{dy_e}{dt}$, which are the x and y components of the speed of the free end of the board, how do we find the composite speed of that end? Velocities that are at right angles to each other combine according to the Pythagorean rule (you can prove that by drawing a diagram that shows the movement of the end of the board in the x and y directions during a short interval of time, Δt, and then taking the limit as Δt goes to zero). So if we allow v to be the composite speed, then

$$v^2 = \left(\frac{dx_e}{dt}\right)^2 + \left(\frac{dy_e}{dt}\right)^2 \tag{8.2-32}$$

Solving for velocity, my calculator tells me that $v = 1.0649110\,\frac{\text{inches}}{\text{sec}}$.

Exercises
Worked solutions are provided at the end of this document or by clicking the links)

1) Return to the taxicab problem (worked example 1), but this time give the westbound taxicab a quarter mile head start. Keep everything else the same. When the northbound taxicab has gone half a mile from its starting position, how fast is the distance between the two taxicabs increasing? You should be able to get through this one using the worked example on this page as a model. When you are done, click here to view the solution.

2) We shall remain in New York City for this problem, which is similar to the taxicab problem. The Verazzano-Narrows bridge, which connects Brooklyn with Staten Island, has a span length of about 4200 feet. Assume that the entire roadbed is 200 feet above the water. A small motor boat traveling perpendicular to the bridge is 1000 feet away from passing under the midspan of the bridge. The boat is cruising 20 feet per second toward the bridge. At the same time a Brooklyn-bound runner on the bridge is passing the tower on the Staten Island side (the towers are at either end of the span, so the runner is entering the span). The runner moves at 10 feet per second along the bridge. How fast is the distance between the boat and the runner decreasing?

Although this one has the additional wrinkle of being in three dimensions instead of two (like the taxicabs), the methods are the same. Remember that the Pythagorean distance formula works just as well in three dimensions as it does in two. So go as far as you can with this before clicking here to see the solution.

3) Here's one that is very important in mechanics. A circular track has radius, r. A train moves around the track at $\omega \frac{\text{radians}}{\text{second}}$. That means that if the center of the circle is at $(0, 0)$, the position of the train as a function of time, t, is:

$$(r\cos(\omega t), r\sin(\omega t)) \tag{8.2-33}$$

Find the x and y components of the train's velocity (which is the first time derivative of position) and the x and y components of its acceleration (which is the second time derivative of position). What can you observe about the direction of the train's velocity and acceleration in relation to its position on the track? And how would the magnitude of velocity and acceleration be effected if you varied r? or if you varied ω? Click here to view solution.

4) A rocket is launched vertically and travels at 100 meters per second. A tracking radar is 500 meters from the launch site. When the rocket is 800 meters high, how fast must the radar antenna slew (in radians per second) in order to track the rocket? Click here to view solution.

5) Two sticks, one 8 feet long, the other 12 feet long, are joined end-to-end with a hinge. The free end of the 8 foot stick is fixed to the ground. The free end of the 12 foot stick is slid along the ground toward the free end of the 8 foot stick at one foot per second. The whole assembly is held so that the triangle formed by the two sticks and the ground is always perpendicular to the ground. At what rate is the angle between the 8 foot stick and the ground increasing when the distance between the free ends of the two sticks is 5 feet? **Hint:** Use the law of cosines. When you've worked this one out, click here to view solution.

6) The celebrated actor, Tyrone Olivier, is exactly 6 feet tall. Tonight he delivers a stirring soliloquy on stage. For dramatic effect he is illuminated only by a single footlight, which is at the same level as the floor of the stage. 24 feet behind the footlight is the vertical backdrop of the stage. As Tyrone Olivier nears the climax of his soliloquy, he begins walking toward the audience (and toward the footlight) at 2 feet per second. When he is 8 feet from the footlight, at what rate is his shadow on the backdrop increasing in height? **Hint:** Make a diagram and use similar triangles. Click here for solution.

Dimensional Checking (sometimes called Units Checking)

For another point of view on this topic by R. Horan and M. Lavelle, click here

Does Your Answer Pass the Smell Test?

In the early paragraphs of this section I suggested that you always do reasonableness checks on your answers to problems like these. The tests I suggested were things like making sure that the sign of the answer made sense and that the answer seemed in the right ballpark. But till now I have completely skipped the most powerful method of checking your answer, and that is checking it dimensionally.

If you have taken or are taking chemistry or physics, you are likely to have already encountered this method. Besides working on chemistry and physics problems, it works on related rate problems as well (and just about all other word problems too). If you are going into engineering, this method will serve you well for many years to come.

Back in grade school when you learned to do word problems, your teacher, if he or she was any good, told you, "Always write in the units." This was and still is excellent advice. In the real world most quantities have units like meters, dollars, pints, degrees C, and so on. If you drive 60 miles per hour, you are also driving 88 feet per second. So if you had to write down how fast you were going, which is correct, 60 or 88? Neither! You could be indicating leagues per sidereal day for all I know. But both "60 miles per hour" and "88 feet per second" are correct answers. And the two quantities are indeed equal, despite the difference in the numerals. This is because the "miles per hour" and the "feet per second" are as much a part of each quantity as the 60 or the 88.

The real beauty of having units attached to your quantities is that you can do an arithmetic of sorts on the units themselves. And when you are done, if the problem asks, *"how fast is Jane running?"* the answer better come out having units of velocity. If it doesn't then you did something wrong somewhere and your answer is probably wrong.

Let's take a very simple example. Light travels 299,799 kilometers in a second. What is the speed of light in kilometers per year if there are 31,557,600 seconds in a year?

$$299,799\,\frac{\text{km}}{\cancel{\text{sec}}} \times 31,577,600\,\frac{\cancel{\text{sec}}}{\text{year}} = 9.460937 \times 10^{12}\,\frac{\text{km}}{\text{year}}$$

Remember that whenever you see the word, *"per,"* it means the same as *"divided by"* or *"over."* And as it is with numbers, so it is with units. "Kilometers *per* second" means the same thing as "kilometers over seconds." That is how we wrote it here. Same thing with "seconds in a year." It also means "seconds *per* year," which means "seconds *over* years." The problem asks for an answer in "kilometers *per* year." You can see that when we multiply "kilometers *over* seconds" times "seconds *over* years," the seconds cancel and we are left with "kilometers *over* years," which are the units the problem calls for. Since the units came out right, there is a good likelihood that we did the problem correctly.

Now suppose a problem asked, how many seconds does it take light to travel 1.5×10^8 kilometers? Here is the setup for that:

$$1.5 \times 10^8\,\cancel{\text{km}} \times \frac{1}{299,799}\,\frac{\text{sec}}{\cancel{\text{km}}} = 500\,\text{sec} \qquad (8.2\text{-}34)$$

The problem asks for an answer in seconds. Observe how, in order to get kilometers to cancel and for seconds to come out on top, we had to take the reciprocal of "kilometers per second" and multiply by it. When you take the reciprocal of the units, you also take the reciprocal of the number attached to them. Or equivalently you could have just divided by the kilometers per second quantity. So the units are cluing you in on what gets multiplied and what gets divided to get the correct answer.

Some quantities have no units. These are called *dimensionless* quantities. For example, π is the ratio of the circumference of a circle to its diameter. It doesn't matter how large the circle is or in what units you measure the circumference and diameter (as long as you use the same units for both), the value of π is still the same. If c is the circumference in furlongs and d is the diameter in furlongs, then

$$\frac{c\,\cancel{\text{furlongs}}}{d\,\cancel{\text{furlongs}}} = \pi$$

The units cancel in this case, no matter what units you use.

Angles, and ***particularly angles measured in radians are dimensionless.*** Remember that the definition of radian measure was that you take the distance around the circle the angle takes you, and then you divide that distance by the radius of the circle. That is the quotient of two lengths, so the length-units cancel leaving only the dimensionless ratio.

Units can be raised to a power. You know very well that if you multiply the height of a rectangle by its width, you get its area. If both the height and width are in inches, then the area comes out in *square* inches, which is the same as inches *squared*.

$$14\,\text{in} \times 7\,\text{in} = 98\,\text{in}^2$$

Not only length-units can be raised to a power. It can happen to other types of units as well. For example, acceleration measures how many meters per second an object's velocity gains *per* second. So the units of acceleration would be meters per second per second.

$$\frac{\;\frac{\text{meters}}{\text{sec}}\;}{\text{sec}} = \frac{\text{meters}}{\text{sec}^2}$$

You can see how that's the same as meters per second *squared*.

You can only add quantities that have the same units. In other words, you can't add apples to oranges. So if you find yourself adding two quantities whose units differ, you made a mistake somewhere. The units that the sum has are the same as the units of each of its summands.

When you use trig functions, inverse trig functions, logs or exponentials, the argument must be dimensionless. You can't take the sine of 3 centimeters or the exponential of 40 grams. The thing you take the sine or exponential (or any of the other trig or log functions) of must not have any units. And the value the function returns is also dimensionless. For example, if you have an exponential function of time, the time, t, will always be multiplied by a rate, k, whose dimensions are inverse time. Suppose t is in seconds. Then there would be a rate, k, whose dimensions are in per second (that is inverse seconds). Suppose $k = \frac{0.001}{\text{second}}$. Then you could form the exponential function

$$f(t) = e^{-kt} = e^{-0.001t}$$

Of course this equation gives you a value, $f(t)$, that is dimensionless. If the result had to have dimensions of, say watts, then the exponential would have to have a multiplier that had dimensions of watts. For example, you might have $P = 3 \times 10^7$ watts. Then you could have

$$g(x) = P\, e^{-kt} = 3 \times 10^7 \,\text{watts} \times e^{-0.001t}$$

The only possible exception to this rule is the log. Since a quantity is the product of its units and a number, you can argue that

$$\ln\left(\frac{1400\,\text{cm}}{14\,\text{cm}}\right) = \ln(1400\,\text{cm}) - \ln(14\,\text{cm}) = \ln(1400) - \ln(14) + \cancel{\ln(\text{cm})} - \cancel{\ln(\text{cm})}$$

but only if you are willing to accept the existence of the abstraction, ln (cm), and only if, in the end, such abstractions all cancel out. The log still returns a dimensionless result regardless.

Units conversion factors, deep down, are all equal to one. Consider that

$$1\,\text{hour} = 60\,\text{minutes}$$

That means that if you divide out 1 hour, you get the units conversion factor of

$$1 = 60\,\frac{\text{minutes}}{\text{hour}}$$

When you convert 14 hours to minutes, you do:

$$14\,\cancel{\text{hours}} \times 60\,\frac{\text{minutes}}{\cancel{\text{hour}}} = 840\,\text{minutes}$$

Isn't it true that 14 hours and 840 minutes are precisely the same length of time? The two quantities are equal. Which means that $60\,\frac{\text{minutes}}{\text{hour}}$ must be equal to 1 and nothing else.

A derivative always takes the ratio of units. For example, when you take the derivative of distance (say in meters) with respect to time (say in seconds), the derivative would have units of meters per second. If you have any dependent variable, v, and you take its derivative with respect to an independent variable, t, to get $\frac{dv}{dt}$, the result will have units that are v's units divided by t's units.

Units of second derivatives are tricky. If you take the second derivative of a dependent variable, v, with respect to the independent variable, t, to get

$$\frac{d^2v}{dt^2}$$

this second derivative will have units that are the units of v divided by the *square* of the units of t. Note that you *only* square the denominator units and not the numerator units. This makes sense because the second derivative is the derivative of the first derivative. So you take the units of the first derivative and divide it by the units of the independent variable. For example, acceleration is the second time derivative of distance. If you are using meters and seconds, acceleration has units of

$$\frac{\text{meters}}{\text{sec}^2}$$

This same principle goes with higher derivatives too. If you take the nth derivative of v with respect to t, the units you end up with are the units of v divided by the units of t taken to the nth power. Only the denominator units are taken to the nth power, **not** the units of v.

Let's dimensionally check a solution to one of the problems in this section. We'll do problem 4. Here you were given the rocket's altitude, h in meters, the distance to the radar, x in meters, and the rocket's speed, $\frac{dh}{dt}$ in meters per second. The problem asks for a rate, $\frac{d\theta}{dt}$, in radians per second. Since radians is dimensionless, that is the same as plain old "per second" (when the *"per"* is not preceded by any units, it is assumed that the numerator of the units is dimensionless, which is to say that the numerator is just 1). The solution equation for the problem is

$$\frac{dh}{dt}\frac{x}{x^2+h^2} = \frac{d\theta}{dt}$$

On the left we have $\frac{dh}{dt}$, which is in meters per second. The next factor has x in the numerator, so the numerator is meters. The denominator has the sum of x^2 and h^2. Both x and h have units of meters, so their squares both have units of square meters. Because those units are the same, it is ok to add those two quantities. To the right of the equal sign is $\frac{d\theta}{dt}$, which we already determined has units of per second. So, expressing this equation only in its units we have

$$\frac{\text{meters}}{\text{sec}} \frac{\text{meters}}{\text{meter}^2} = \frac{1}{\text{sec}}$$

I'm sure you can convince yourself that meters and meters squared all cancel out of the left-hand side of this, leaving only per second on both the left and right of the equal. So the units agree. The original equation smells good.

Solutions to Exercises

1) Modified Taxicab Problem: The problem was that a northbound taxicab starts from the intersection of Park Ave. and 59^{th} St. at 40 miles per hour. At the same time a westbound taxicab starts from one quarter mile west of Park and 59^{th} at 30 miles per hour. When the northbound taxicab has gone one half mile from its starting position, how fast is the distance between the two taxicabs increasing?

Recall that in the original taxicab problem you came up with a relationship between y, which was how far north of Park and 59^{th} the northbound taxicab was, and x, which was how far west of Park and 59^{th} the westbound taxicab was. This relationship is the only thing that has changed from the original problem. By giving the westbound taxicab a head start, you have added a fixed amount to x. So, subtracting that fixed amount from x to get how far the westbound taxicab has traveled since the start, you now would have

$$\frac{y}{40 \text{ mph}} = \frac{x - 0.25 \text{ miles}}{30 \text{ mph}}$$

or equivalently

$$\frac{3}{4}y + \frac{1}{4}\text{mile} = x$$

That is, if you subtract the quarter-mile head start from x, then the speeds (30 mph and 40 mph) must govern the relationship between how far the two taxicabs have gone.

Everything else is the same in this problem. You still have the Pythagorean distance formula dictating that:

$$s^2 = x^2 + y^2$$

where s is the distance separating the taxicabs. And you can still take the derivatives of the two relationships:

$$\begin{aligned} \frac{3}{4}\frac{dy}{dt} &= \frac{dx}{dt} \\ 2s\frac{ds}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \end{aligned}$$

and you can still make the substitutions suggested by the relationships. We cancel the 2's in the above. Then using substitutions based upon $s^2 = x^2 + y^2$ and $\frac{dx}{dt} = \frac{3}{4}\frac{dy}{dt}$ we have:

$$\sqrt{x^2 + y^2}\frac{ds}{dt} = \frac{3}{4}x\frac{dy}{dt} + y\frac{dy}{dy}$$

Now replace x with $\frac{3}{4}y + \frac{1}{4}$ mile, multiply things out, and gather like terms:

$$\sqrt{\frac{25}{16}y^2 + \frac{3}{8}y + \frac{1}{16}}\,\frac{ds}{dt} = \left(\frac{25}{16}y + \frac{3}{16}\right)\frac{dy}{dt}$$

Remember that y was given as half a mile, and $\frac{dy}{dt}$ was given as 40 miles per hour. Poking the calculator indicates that the above becomes:

$$0.8003905\,\frac{ds}{dt} = 38.75$$

$$\frac{ds}{dt} = 48.4138662 \text{ mph}$$

2) The 3-Dimensional Taxicab Problem: The problem was on the Verazanno-Narrows bridge, whose span is 4200 feet and whose road-height is 200 feet. You have a boat approaching midspan 1000 feet away at the same time as a runner enters the span. The boat goes 20 feet per second, the runner goes 10 feet per second. How fast is the distance separating them decreasing?

Quicly going through the first few of our seven questions: the independent variable is again time. For dependent variables, you have to set up a coordinate system. There are several alternatives that will work, and I will pick just one (you may have used a different one and still come up with a correct answer). I pick the origin as the point on the water directly under midspan. The bridge is parallel to the x-axis, and the boat is traveling along the y-axis. That gives us the dependent variable, y, as the boat's position, and it gives us the dependent variable, x, as the runner's position. The z coordinates of both the runner and the boat are constant (the runner is at $z = 200$ ft, and the boat is at $z = 0$ ft).

Now we get to question 4, which is to come up with the relationships among the variables. The Pythagorean distance formula is one such relationship. It gives you

$$s^2 = x^2 + y^2 + 200^2$$

where s is the distance separating them. As for the other relationship, you have to expand what you did in the second taxicab problem when you gave the westbound taxicab that head start.

Let's say that the x-axis of our coordinate system puts Staten Island on the negative side and Brooklyn on the positive side. Then the runner starts out in negative x territory and runs toward $x = 0$. Let's also have the boat start in negative y territory and motor toward $y = 0$.

In this problem neither the runner nor the boat have head starts. They both have hind starts (starting in back of the origin instead of in front of it). Recall that when you gave the westbound taxicab a head start you subtracted its head start from its east-west position when you made up the relationship between that and its north-south position. Likewise, when there is a hind start, you will add it to the position along the appropriate axis. Hence:

$$\frac{x + 2100}{10} = \frac{y + 1000}{20}$$

The 2100 comes from the span being 4200 feet and midspan being where $x = 0$. So 2100 feet is the length of the hind start given the runner. From the above you can use elementary algebra to get

$$2x + 3200 = y$$

Question 5 asks you to take the derivatives of the relationships. With the Pythagorean relationship you get

$$2s\,\frac{ds}{dt} = 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt}$$

from which you can cancel the factor of 2. And the other relationship gives

$$2\,\frac{dx}{dt} = \frac{dy}{dt}$$

which just says that the boat's speed is twice the runner's speed, and we already knew that.

Question 6 asks you to consider what the problem is asking for. The separation distance is s, and the problem is asking how fast that distance changes, which is $\frac{ds}{dt}$. So that is what you have to solve for. You have an equation above that has $\frac{ds}{dt}$ in it, and you have another equation that suggests a way to substitute for s. So you have:

$$\sqrt{x^2 + y^2 + 200^2}\,\frac{ds}{dt} = x\,\frac{dx}{dt} + y\,\frac{dy}{dt}$$

When you ask yourself question 7, you find that the problem gives you numbers for x, y, $\frac{dx}{dt}$, and $\frac{dy}{dt}$. The way we set up the coordinates we have:

$$\begin{aligned}
x &= -2100 \text{ feet} \\
y &= -1000 \text{ feet} \\
\frac{dx}{dt} &= 10 \ \tfrac{\text{ft}}{\text{sec}} \\
\frac{dy}{dt} &= 20 \ \tfrac{\text{ft}}{\text{sec}}
\end{aligned}$$

Putting all those numbers into the last equation and solving for $\frac{ds}{dt}$ you should get

$$\begin{aligned}
s &= 2334.5235 \text{ feet} \\
\frac{ds}{dt} &= -17.56247 \ \tfrac{\text{ft}}{\text{sec}}
\end{aligned}$$

Observe that $\frac{ds}{dt}$ is negative, indicating that the distance separating the runner and the boat, s is decreasing. That is as expected. We also expect that the distance should be closing at a rate that is *in the vicinity* of the 10 or 20 feet per second with which the objects are moving, so you can see that the answer is in the right ballpark (if you had come up with 100 feet per second or 0.1 feet per second you would have had to view your answer with suspicion).

3) Circular Motion: This one is barely even a related rate problem. It is more a problem of finding derivatives. But if you are going into engineering, the relationships that flow out of this problem will have to become second nature to you.

An object (could be a train or anything else) traveling around a circular path of radius r centered at the origin is at point:

$$(r\cos(\omega t), r\sin(\omega t))$$

at time t seconds if its angular velocity is $\omega \ \frac{\text{radians}}{\text{second}}$. We need to take first and second derivatives of the expressions above to find the x and y components of the object's velocity and acceleration.

Taking the first time derivative of each of the components of the position described above gives:

$$(-r\omega\sin(\omega t), r\omega\cos(\omega t))$$

So when the object is in the first quadrant, for example, its x velocity component is negative – that is its net x motion there is *toward* the y-axis. Its y velocity component is positive in the first quadrant, so its net y velocity there is *away* from the x-axis.

But most important is that the velocity vector is *at right angles* to the position vector. To confirm this, pick any point, (x, y). Plot it on graph paper and draw a line connecting it to the origin. Now plot $(-y, x)$ and draw a line connecting it to the origin. No matter what x and y you chose, the two lines you drew were at right angles to each other. Indeed, you could have *scaled* the second point by some value, k (which simply means multiply both the x and y components by k), and the lines would still have been at right angles to each other. Do you see how the velocity components relate to the position components in this problem in exactly the same way?

Now lets find the acceleration by taking the derivative of the velocity:

$$\left(-r\omega^2 \cos(\omega t), -r\omega^2 \sin(\omega t)\right)$$

Observe that the acceleration points in *exactly the opposite direction* as the position. That is the train is accelerated toward the center of the circle. You have surely noticed that when a car accelerates forward you feel a reaction pressing you back into your seat – that is you feel a reaction opposite the acceleration. Here as the train goes around the circle it is accelerated *inward*, and somebody aboard the train would feel a reaction *outward*. The outward reaction is what is commonly referred to as *centrifugal force*.

Finally, ω is the angular rate in radians per second that the train goes around the circle. If you wanted to convert it to revolutions per second you would divide ω by 2π. If you double ω, you should be able to see that you consequently double the magnitude of the velocity. It should be equally clear that you would *quadruple* the magnitude of the acceleration, since acceleration is shown to be proportional to the *square* of the angular rate, ω. This also means that the centrifugal force increases with the *square* of ω.

For your own amusement, see what happens if the circle is not centered at the origin. Suppose it is centered at (x_0, y_0). Then the position is given by

$$(r\cos(\omega t) + x_0, r\sin(\omega t) + y_0)$$

Remember that x_0 and y_0 are both constants. What effect does this have on the velocity and acceleration of the train?

How about if the train does not start at the angular position of zero radians? Suppose it starts at ϕ radians. Then its position is given by

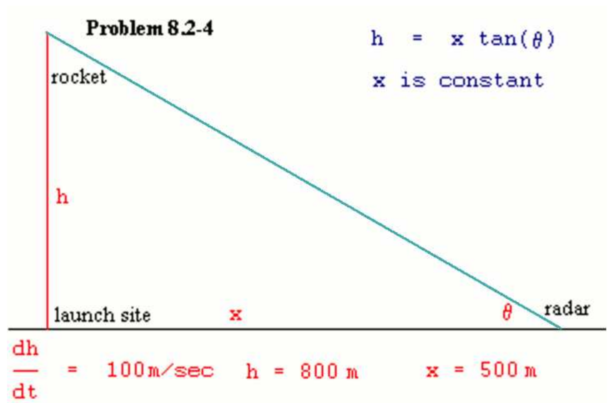
$$(r\cos(\omega t + \phi), r\sin(\omega t + \phi))$$

What effect does ϕ have on the magnitude of the velocity and the acceleration? Remember that $\sin^2(x) + \cos^2(x) = 1$ for all x.

4) The Rocket Problem: The problem is: A rocket is launched vertically and travels at 100 meters per second. A tracking radar is 500 meters from the launch site. When the rocket is 800 meters high, how fast must the radar antenna slew (in radians per second) in order to track the rocket?

You should be getting pretty good at answering questions 1 and 2 by now. Is it clear to you that time, t, is the independent variables in this problem and that rocket height, h, and and the angle that the radar looks up, θ, are the dependent variables? And since the radar does not get any nearer or farther from the launch site, it should be clear too that the distance, x, between the radar and the launch site is constant.

In this problem it pays to draw a diagram (did you?). Here is my diagram of the problem (not drawn to scale).



Passing quickly to question 3, there are no hidden variables here. All of them are shown in the diagram.

As to question 4, the relationship between the two dependent variables, h and θ, is gathered by analyzing the right triangle in the diagram. Since $\tan(\theta)$ is opposite over adjacent, it follows that

$$\tan(\theta) = \frac{h}{x}$$

and

$$x \tan(\theta) = h$$

Question 5 asks you to take the derivative of each relationship you have among dependent variables. Remembering that x is a constant here and h and ? are dependent variables, you have

$$x \sec^2(\theta) \frac{d\theta}{dt} = \frac{dh}{dt}$$

Question 6 asks you to figure out what the problem is asking for and solve it. Since θ is the radar's "look" angle and the problem is asking how fast the radar must slew, it should be clear that the problem is asking you for $\frac{d\theta}{dt}$. Solving the above for that gives:

$$\frac{1}{x}\frac{dh}{dt}\cos^2(\theta) = \frac{d\theta}{dt}$$

But you can make this even simpler by remembering that

$$\sec^2(\theta) = 1 + \tan^2(\theta)$$

and that

$$\tan(\theta) = \frac{h}{x}$$

If you substitute these facts before you solve for $\frac{d\theta}{dt}$, you find that

$$\frac{1}{x}\frac{dh}{dt}\frac{1}{1+\frac{h^2}{x^2}} = \frac{d\theta}{dt}$$

This looks rather nasty, but multiply out the x^2, then cancel the $\frac{1}{x}$ and you have

$$\frac{dh}{dt}\frac{x}{x^2+h^2} = \frac{d\theta}{dt}$$

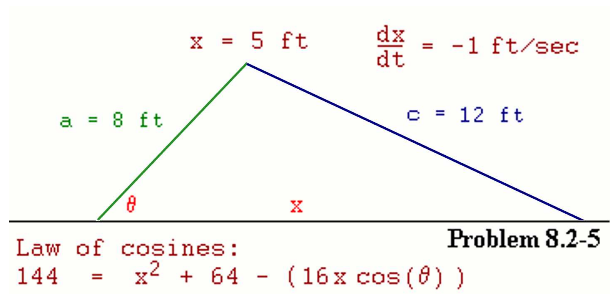
Question 7 has you plug in the numbers. We have $x = 500$ meters, $h = 800$ meters, and $\frac{dh}{dt} = 100\ \frac{\text{meters}}{\text{second}}$. When I put them into the solution-equation, I get

$$\frac{d\theta}{dt} = 0.056179775\ \frac{\text{radians}}{\text{sec}} = 3.218864018\ \frac{\text{degrees}}{\text{sec}}$$

What happens to the radar's slew rate as the rocket gets higher according to the solution equation above? Is that what you'd expect it to do based upon your intuition about tracking a rocket in a vertical trajectory?

Pair-of-sticks Problem: The problem was: Two sticks, one 8 feet long, the other 12 feet long, are joined end-to-end with a hinge. The free end of the 8 foot stick is fixed to the ground. The free end of the 12 foot stick is slid along the ground toward the free end of the 8 foot stick at one foot per second. The whole assembly is held so that the triangle formed by the two sticks and the ground is always perpendicular to the ground. At what rate is the angle between the 8 foot stick and the ground increasing when the distance between the free ends of the two sticks is 5 feet? The hint was to use the law of cosines.

The first thing to do here is make a diagram. Here's mine. If you made a diagram, you probably got something like this, but you probably didn't do it in color. Here the 8 foot stick is shown in green, the 12 foot stick in blue. The distance between the far ends of the stick (that is along the ground) is x. The angle the problem talks about is θ. Again the diagram is not drawn to scale, and your diagram need not be drawn to scale either in order for it to be useful in tackling this problem. Besides, without the aid of a compass, it would be hard to draw this one to scale because we do not yet know the value of θ.



Answer to question 1 again is that time, t, is the independent variable. And to question 2, clearly x and θ are the dependent variables. And to question 3, there are no hidden variables.

And now to question 4 – what is the relationship between x and θ? This is given by the law of cosines. When you plug in the parameters of this problem, you get

$$c^2 = x^2 + a^2 - 2ax\cos(\theta) \qquad \text{(p5.1a)}$$

or in numbers, as is shown in the diagram:

$$144 = x^2 + 64 - 16x\cos(\theta) \qquad \text{(p5.1b)}$$

Now comes question 5, and things get just a little dicey. You do have to use the product rule in order to take the derivative of the right-hand term of equation p5.1a.

$$0 = 2x\,\frac{dx}{dt} + 2ax\sin(\theta)\,\frac{d\theta}{dt} - 2a\cos(\theta)\,\frac{dx}{dt} \qquad \text{(p5.2)}$$

error(1): wrong syntaxThe problem asks for $\frac{d\theta}{dt}$, so in answering question 6, you need to figure out a way to solve for that based upon what the problem gives us, which is a value for x and a value for $\frac{dx}{dt}$. The difficulty, of course, is that your solution must involve $\cos(\theta)$ and $\sin(\theta)$. But using the law of cosines equation (eq. p5.1a), you can solve for $\cos(\theta)$.

$$\cos(\theta) = \frac{a^2 + a^2 - c^2}{2ax} \tag{p5.3}$$

and from that you can readily find $\sin(\theta)$ using the identity:

$$|\sin(\theta)| = \sqrt{1 - \cos^2(\theta)} \tag{p5.4}$$

Since θ can be no more than π radians (180 degrees), we are guaranteed that $\sin(\theta)$ will be positive, and so we can drop the absolute value in the above.

When you solve eq. p5.2 for $\frac{d\theta}{dt}$ you get

$$\frac{d\theta}{dt} = \frac{2a\cos(\theta) - 2x}{2ax\sin(\theta)}\frac{dx}{dt} \tag{p5.5}$$

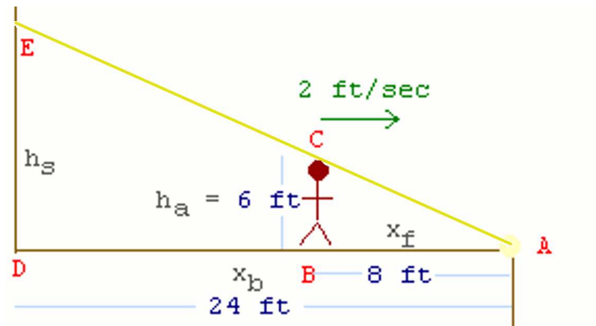
Finally to question 7 – putting in the numbers. The problem gives us that $x = 5$. From the diagram we have $a = 8$ and $c = 12$. From this equation, p5.3 gives you that $\cos(\theta) = -55/80 = -0.6875$. From that and equation p5.4 you have that $\sin(\theta) = 0.726184377$. We know that x is decreasing at a rate of 1 foot per second, so $\frac{dx}{dt} = -1$. Putting that all into equation p5.5 you have

$$\frac{d\theta}{dt} = 0.361478445\ \frac{\text{radians}}{\text{second}} = 20.71118932\ \frac{\text{degrees}}{\text{second}}$$

Note that the rate is positive, so θ is increasing. Does that seem reasonable if x is decreasing? Refer back to the diagram.

Note also that starting with equation p5.1b you could have carried numerical values down into the subsequent equations instead of the symbols, a and c. If you made no mistakes you would have ended up with the same answer. In general, though, it is not a good idea to do this. If you carry the symbols down, reaching a fully symbolic solution right up until the last steps (e.g., equation p5.5), not only are you immune to a greate many arithmetic mistakes you might have made otherwise, but you also have an opportunity to do dimensional checking on your solution.

6) The Footlight Problem: The problem was: Actor, Tyrone Olivier, is exactly 6 feet tall. He delivers a soliloquy on stage, illuminated only by a single footlight, which is at the same level as the floor of the stage. 24 feet behind the footlight is the vertical backdrop of the stage. He begins walking toward the footlight at 2 feet per second. When he is 8 feet from the footlight, at what rate is his shadow on the backdrop increasing in height? The hint was to make a diagram and use similar triangles. Here's the diagram. Observe that triangle **ABC** is similar to triangle **ADE**. This means that the ratio of lengths, $\overline{\mathbf{AB}}$: $\overline{\mathbf{BC}}$ is the same as the ratio of lengths $\overline{\mathbf{AD}}$: $\overline{\mathbf{DE}}$. Note also that the lengths of $\overline{\mathbf{AB}}$, $\overline{\mathbf{BC}}$, and $\overline{\mathbf{AD}}$ are all given in the problem (although length $\overline{\mathbf{AB}}$ does vary with time). With that in mind, let's go and answer the seven questions.



Problem 8.2-6: The Footlight and the actor

Answering question 1, we can see that the problem asks for a rate (in time) at which the actor's shadow is growing. So the independent variable is again time, t.

Answering question 2, since the problem asks for the rate at which the shadow height is growing, shadow height, h_s, seems an obvious candidate as a dependent variable. That height is the same as length $\overline{\mathbf{DE}}$. And since the problem talks about his walking toward the footlight, his distance from the footlight, x_f, also seems an obvious candidate. That distances is the same as length $\overline{\mathbf{AB}}$. Note that the height of the actor, h_a, and the distance from the footlight to the backdrop, x_b, are both constants. These correspond to lengths $\overline{\mathbf{BC}}$ and $\overline{\mathbf{AD}}$ respectively.

Answering question 3, there are no inferred variables in this problem. The dependent variables you came up with answering question 2 covers everything there is about this problem.

nswering question 4, the relationship between the two dependent variables, x_f and h_s, are determined by having similar triangles. A little while back you saw that the ratio, $\overline{\mathbf{AB}}$: $\overline{\mathbf{BC}}$, had to be the same as the ratio, $\overline{\mathbf{AD}}$: $\overline{\mathbf{DE}}$. Isn't this the same as the equation,

$$\frac{x_f}{h_a} = \frac{x_b}{h_s} \tag{p6.1}$$

Question 5 tells you to take the derivative with respect to the independent variable of whatever relationship(s) you came up with when you answered question 4. Note that in equation p6.1, the symbols, x_b and ha are both constants (the backdrop does not get any nearer or farther from the footlight, and the actor himself does not grow or shrink). Only x_f and h_s are functions of the independent variable, t. So they are the only ones you take derivatives of. You must treat the constants as scalars (that is they are just along for the ride).

$$\frac{1}{h_a}\frac{dx_f}{dt} = -\frac{x_b}{h_s^2}\frac{dh_s}{dt} \tag{p6.2}$$

Question 6 tells you to solve for whatever the problem is asking for. In this case it is asking for the shadow height the rate at which the shadow grows, which is $\frac{dh_s}{dt}$.

$$-\frac{h_s^2}{h_a x_b}\frac{dx_f}{dt} = \frac{dh_s}{dt} \tag{p6.3}$$

Question 7 tells you to put in the numbers the problem gives you. These are $x_f = 8$ ft, $h_a = 6$ ft, and $x_b = 24$ ft for distances. The problem also tells you that the actor is moving toward the footlight at 2 $\frac{\text{ft}}{\text{sec}}$. This means that x_f is decreasing at that rate. So you know that $\frac{dx_f}{dt} = -2\ \frac{\text{ft}}{\text{sec}}$ (make sure you see why this derivative is negative). Plugging all this into equation p6.3 gives

$$-\frac{h_s^2}{144}(-2) = \frac{h_s^2}{72} = \frac{dh_s}{dt} \tag{p6.4}$$

The only difficulty is that you cannot yet plug in a value for h_s. But fortunately equation p6.1 gives you a way to solve for it.

$$h_s = \frac{x_b}{x_f}h_a = \frac{24}{8} \times 6 = 18 \text{ ft} \tag{p6.5}$$

So in the end equation p6.4 becomes

$$\frac{dh_s}{dt} = \frac{18^2}{72} = \frac{324}{72} = 4.5\ \frac{\text{ft}}{\text{sec}} \tag{p6.6}$$