Solving Cubic and Quartic Polynomials

Karl Hahn

July 23, 2005

1 Solving Cubic Polynomials

If you are given a cubic equation in the form of $x^3+px^2+qx+r=0$ and need to solve for x, then the first thing you do is substitute variables. Everywhere you see an x in the cubic, replace it with

$$x = u - \frac{p}{3} \tag{1}$$

When you get done squaring and cubing this expression, then substituting stuff back in and gathering like terms, you will get

$$u^3 + au + b = 0 \tag{2}$$

where

$$a = q - \frac{p^2}{3} \tag{3}$$

and

$$b = r - \frac{pq}{3} + \frac{2p^3}{27} \tag{4}$$

Now compute A and B by

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \tag{5}$$

and

$$B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \tag{6}$$

Then you have the following solutions for u:

$$u = A + B \tag{7}$$

$$u = -\frac{1}{2}(A+B) + \sqrt{-\frac{3}{4}}(A-B) \tag{8}$$

$$u = -\frac{1}{2}(A+B) - \sqrt{-\frac{3}{4}}(A-B) \tag{9}$$

Of course, you know how convert from the u solutions to the x solutions.

You may be troubled by the expression, $\sqrt{\frac{-3}{4}}$, which is not a real number, but is a *complex number*. Note that A and B will also nonreal complex numbers whenever

$$\frac{b^2}{4} + \frac{a^3}{27} < 0 \tag{10}$$

So the cubic formula does require that you understand the arithmetic of complex numbers.

2 How to Derive the Cubic Formula

If you have been trying to come up with a solution to the cubic on your own, my heart goes out to you. Generations of mathematicians searched for a cubic solution before Niccolo Fontana Tartaglia and Girolamo Cardano hit on it in the 16th century (Cardano published the solution in *Ars Magna* in 1545). So there is no shame in your not finding it (of course if you did find it on your own, my hat is off to you).

Since you solve the quadratic by completing the square, a lot of people who attack the cubic do so by trying to “complete the cube.” Well that attack doesn’t work. The trick is to convert the cubic, by a circuitous route, to a quadratic and then apply the quadratic formula.

We already saw in the previous paragraphs that it is sufficient to find a solution only to cubics in the form of $u^3 + au + b = 0$ This is because by suitable substitution of variables, any cubic can be brought into this form.

In order to find the solution to the cubic you would have had to have the insight of making the simplification of equation 2, and you would also have to have the insight to suppose that a solution to that equation, u, ought to be expressed as the difference of two new variables: $u = s - t$. Now substitute that for u into the cubic and you get $(s-t)^3 + a(s-t) + b = 0$. Now multiply out the cubed term using the binomial formula:

$$s^3 - 3s^2t + 3st^2 - t^3 + as - at + b = 0 \tag{11}$$

In order for this equation to hold we must be able to cancel all the terms on the left of the equal. If you substitute $t^3 - s^3 = b$, you can see that you cancel all the cubed terms:

$$s^3 - 3s^2t + 3st^2 - t^3 + as - at + t^3 - s^3 = 0 \tag{12}$$

which reduces to $-3s^2t + 3st^2 + as - at = 0$. Now just substitute $3st = a$, and you cancel the remaining terms:

$$-3s^2t + 3st^2 + (3st)s - (3st)t = 0 \tag{13}$$

The only problem that remains is to find a suitable s and t from a and b. We already made the substitutions, $3st = a$ and $t^3 - s^3 = b$. So we solve these two equations simultaneously for s and t.

$$s = \frac{a}{3t} \tag{14}$$

Substitute this into $t^3 - s^3 = b$ and you get

$$t^3 - \frac{a^3}{(3t)^3} = b \tag{15}$$

Multiply through by $(3t)^3 = 27t^3$, and it becomes:

$$27t^6 - a^3 = 27t^3b \tag{16}$$

Don't let the 6th power here scare you. This isn't as bad as it looks. All the powers of t are multiples of 3. So if substitute $t^3 = z$, you find yourself in familiar territory.

$$27z^2 - a^3 = 27bz \tag{17}$$

From here you use the quadratic formula to solve for z. For each solution you come up with for z, you will know that $t = \sqrt[3]{z}$. From t you can find $s = \frac{a}{3t}$, and from s you can find $u = s - t$, which is a solution to $u^3 + au + b = 0$, and from u you can find x.

3 Solving Fourth Degree Polynomials (Quartics)

To be honest, I don't think I could have come up with a solution to the quartic on my own even if I had studied the problem for years. What I will do here is reconstruct, as best I can, Ferrari's thinking when he discovered the solution.

The general quartic equation is $x^4 + px^3 + qx^2 + rx + s = 0$. Just as with the cubic, it behooves us to get rid of the second-highest term (in this case the cubed term) by making a substitution of variables. Ferrari probably thought to do this because he already knew the solution to the cubic, which requires a similar simplification, when he solved the quartic. The substitution you make is

$$x = u - \frac{p}{4} \tag{18}$$

When you multiply it out you will get something in the form of

$$u^4 + au^2 + bu + c = 0 \tag{19}$$

which is the *depressed* quartic equation. I will let you multiply it out for yourself to see how a, b, and c relate to p, q, r, and s.

The next thing that is likely to have crossed Ferrari's mind is, "If only I could factor this into two quadratics, then I'd have it nailed." For convenience he made the squared term of each quadratic be u^2 by itself (i.e. no coefficient), since you can always divide out the leading coefficient of any polynomial. So let the product of the two quadratics be $(u^2 + \alpha u + \beta)(u^2 + \gamma u + \delta) = 0$. The next thing Ferrari had to do was contrive a way so that this product would be the same as the depressed quartic [equation 19]. This means finding expressions for α, β, γ, and δ so that when you multiply the two quadratics out you get the depressed quartic.

One thing is perfectly clear. When you multiply out the quadratics, all the cubed terms must cancel. Why? Because the depressed quartic has no cubed terms. When you multiply the two quadratics, the only cubed terms are αu^3 and γu^3 (multiply it out for yourself if you can't see this). To get those two to cancel requires that $\gamma = -\alpha$. This way Ferrari was able to eliminate one variable from the product, $(u^2 + \alpha u + \beta)(u^2 - \alpha u + \delta) = 0$.

The next thing Ferrari probably focused on was how to get all the u^2 terms in the product to add up to au^2 (which is the u^2 term in the depressed quartic [equation 19]). When you gather all the u^2 terms in the product, this requirement becomes:

$$-\alpha^2 u^2 + \beta u^2 + \delta u^2 = au^2 \tag{20}$$

and factoring out the u^2

$$-\alpha^2 + \beta + \delta = a \tag{21}$$

Making this happen is a little tricky. It seems that Ferrari had these thoughts about it:

Let β and δ each contribute half of the a that shows on the right of the above sum. Let them each cancel half of the $-\alpha^2$ term as well. Then it will work out just right. But that would make β equal to δ, unless I add some quantity to one and subtract that same quantity from the other. If I named that quantity, $\frac{1}{2}\rho$, it would would mean

$$\begin{aligned}\beta &= \frac{1}{2}(a+\alpha^2+\rho) \qquad (22)\\ \delta &= \frac{1}{2}(a+\alpha^2-\rho) \qquad (23)\end{aligned}$$

If you substitute this β and δ back into $-\alpha^2+\beta+\delta = a$, you will see that it works no matter what value you choose for ρ. So with the substitutions, the product of quadratics is now:

$$(u^2+\alpha u+\frac{1}{2}(a+\alpha^2+\rho))(u^2-\alpha u+\frac{1}{2}(a+\alpha^2-\rho)) = 0 \qquad (24)$$

Now comes the next hurdle Ferrari had to get over. That is making all the linear u terms (that is terms with no exponent on the u) come out right. When you gather them up from the product given in the above equation, they have to add up to bu in order to match the depressed quartic [equation 19]. This means that

$$\frac{1}{2}\alpha u(a+\alpha^2-\rho) - \frac{1}{2}\alpha u(a+\alpha^2+\rho) = bu \qquad (25)$$

When you take the cancelations and solve what's left for ρ you get

$$\rho = -\frac{b}{\alpha} \qquad (26)$$

which you substitute back into the equation 25.

The final hurdle for Ferrari was to find the α that makes the constant term come out right. To match the constant term in the depressed quartic [equation 19], the constant term from the product in equation 24 has to be equal to c. That means that

$$\frac{1}{4}(a+\alpha^2+\rho)(a+\alpha^2-\rho) = c \qquad (27)$$

If you multiply through by 4 and then multiply out the difference of squares, you get

$$a^2+2a\alpha^2+\alpha^4-\rho^2 = 4c \qquad (28)$$

Now plug in the solution you got for ρ in equation 26.

$$a^2 + 2a\alpha^2 + a^4 - \frac{b^2}{\alpha^2} - 4c = 0 \tag{29}$$

Multiply through by α^2

$$a^2\alpha^2 + 2a\alpha^4 + \alpha^6 - b^2 - 4c\alpha^2 = 0 \tag{30}$$

We have to solve for α in order to find the all the coefficients of the two quadratics. And here α is expressed as the solution to a 6^{th} degree polynomial. But things aren't as bad as they look. All the exponents of α are even. So substitute $z = \alpha^2$, and now you have the cubic polynomial:

$$\begin{aligned} a^2z + 2az^2 + z^3 - b^2 - 4cz &= 0 \quad (31) \\ z^3 + 2az^2 + (a^2 - 4c)z - b^2 &= 0 \quad (32) \end{aligned}$$

Now it's all downhill. Use the cubic formula to solve for z. Take the $\sqrt{z}$ to get α. Then use α to find the coefficients of the two quadratics. Then use the quadratic formula on each of them to come up with solutions for u. Then use equation 18 on your u's to find your solutions for x.