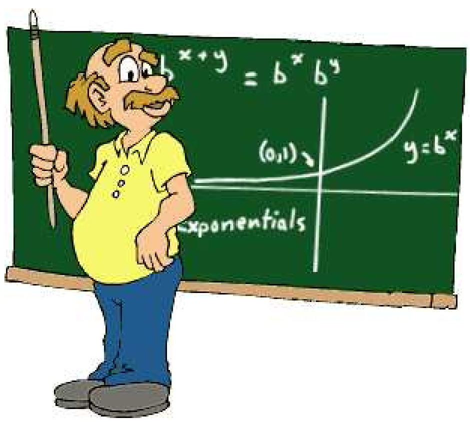
6.1 Be Fruitful and Multiply (intro to Exponentials)



Be fruitful and multiply. That's what the Lord told Adam and Eve on the sixth day of creation according to Genesis 1:28. Again in Genesis 22:17, the Lord tells Abraham, "I will multiply thy seed as the stars of the heaven, and as the sand which is upon the seashore."

Clearly as far back as Abraham, ancient peoples had some understanding of exponential functions. And Abraham lived ever so long ago.

There is also this legend – not quite as old as Genesis – that by some brave deed a peasant saved all of Persia from its enemies. So pleased was the shah that he offered this peasant a piece of the kingdom. The peasant replied by saying, "Sire, my wants are modest. Simply give me a single grain of wheat for the first square of the chess board, two grains for the second square, four grains for the third, and so on, doubling the number of grains for each square until you reach the sixty-fourth square."

The shah agreed to this in front of witnesses and ordered the court accountant to compute how many sacks of grain that would be. Much to the shah's horror, there weren't that many sacks in all of Persia, much less that much grain. It turns out that if you allow 50 milligrams per grain of wheat, it would take 460 billion tons of wheat (in the ballpark of United States production over a thousand years) to meet the shah's obligation to the peasant.

For his ignorance of exponentials, the shah was embarrassed before his subjects. Whether the peasant was ignorant or not of exponentials, we shall never know. He most certainly paid with his head for his impertinence.

One final example of an exponential. A certain species of bacteria has a generation time of 60 minutes. That means that a bacterium of this species allowed to exist under favorable conditions will, after 60 minutes, divide into two. 60 minutes later those two will divide into four. Another 60 minutes and those four will divide into eight, and so on.

But here is an extra wrinkle to the example. The generation time is not always *exactly* 60 minutes for every bacterium. Sometimes it can be as short as 50 minutes or as long as 70 minutes. It only *averages* 60 minutes. So even if you start out with the original bacterium dividing on the hour, soon some of its descendants will be dividing not just on the hour, but at all times throughout the hour.

We would still expect that after, say, 10 hours, we would find very nearly 1024 (that is 2^{10}) bacteria cells, and at 11 hours we would expect to find very nearly 2048 (that is 2^{11}) bacteria cells. But at 10 hours and 30 minutes, we would expect to find some intermediate number of bacteria cells. What's more, over the course of the hour that elapses from 10 hours to 11 hours, we expect to see 1024 cell divisions, or, on average, about 17 each minute.

We know from the fix the shah found himself in that within a few days the number of bacteria cells will grow very large indeed. And however many there are at any hour, there will be very nearly that many cell divisions over the subsequent hour. So within a few days there will be so many cell divisions each second that you could almost regard the cell divisions as being a continuous process. Any interval of time during which no cell division took place would be so short as to be imperceptible even by the most sensitive of time-measuring equipment.

And that is where we're going with the concept of exponentials. We want to turn them into continuous functions. We begin with the idea of raising a number to some nth power, which we do by multiplying it by itself n times. Whenever n is a counting number, we can always do this. But what about all those real numbers between n and $n+1$? How do we raise a number to those powers? How do we make exponentials into something continuous that we can do calculus on?

Let's begin with the most elementary property of raising some number, b, to a power. If m and n are counting numbers, then we have by definition,

$$b^n = b \times b \times \ldots \; n \text{ times } \ldots \times b \qquad (6.1\text{-}1a)$$

$$b^m = b \times b \times \ldots \; m \text{ times } \ldots \times b \qquad (6.1\text{-}1b)$$

From that it is quite clear that

$$b^{m+n} = b \times b \times \ldots\ m+n \text{ times } \ldots \times b \ = \\ (b \times b \times \ldots\ m \text{ times } \ldots \times b) \times (b \times b \times \ldots\ n \text{ times } \ldots \times b) \qquad (6.1\text{-}2)$$

We infer from this the general rule (which you already know) that

$$b^{m+n} \ = \ b^m \ \times \ b^n \qquad (6.1\text{-}3)$$

In words this means that to take the product of some number raised to two powers, find the sum of the exponents and raise it to that power.

This rule forms the foundation of our understanding of exponentials. As we develop exponentials into continuous functions, we shall keep falling back on this rule to guide our way. That is, we will want our extended definitions of exponentials to obey the rule just as our original definition did when we restricted exponents to being counting numbers.

For example, if we want to discover what b^0 is, then we simply look at what happens to the rule when we take

$$b^n \ = \ b^{n+0} \ = \ b^n \times b^0 \qquad (6.1\text{-}3a)$$

Dividing both the left expression and the right expression by b^n we get

$$1 \ = \ \frac{b^n}{b^n} \ = \ \frac{b^{n+0}}{b^n} \ = \ b^0 \qquad (6.1\text{-}3b)$$

This works for any b except zero, so we shall leave aside the issue of raising zero to powers that are not counting numbers.

Likewise the rule leads us to a method on raising numbers to a negative power. If n is a counting number, then by the rule we have:

$$1 \ = \ b^0 \ = \ b^{n+(-n)} \ = \ b^n \times b^{-n} \qquad (6.1\text{-}4a)$$

Dividing through by b^n we have:

$$\frac{1}{b^n} \ = \ \frac{b^0}{b^n} \ = \ \frac{b^{n+(-n)}}{b^n} \ = \ b^{-n} \qquad (6.1\text{-}4b)$$

That takes care of raising any nonzero b to any power that is an integer. But what about all those real numbers between the integers? How does the rule help us with them?

To deal with them we observe another consequence of the rule. If p and q are counting numbers then we have from basic arithmetic:

$$pq \ = \ p + p + \ldots\ q \text{ times } \ldots + p \qquad (6.1\text{-}5)$$

So it must also be true that

$$b^{pq} \ = \ b^{\left(p+p+\ \ldots\ q \text{ times } \ldots\ +p\right)} \qquad (6.1\text{-}6a)$$

If you apply the rule to the right-hand expression in this, you get:

$$b^{pq} \;=\; b^p \times b^p \times \;\ldots\; q \text{ times } \ldots \times b^p \tag{6.1-6b}$$

And by the original definition of how to raise a number to a power that is a counting number, the right-hand side of the above is the same as $(b^p)^q$. So, as a rule, when p and q are counting numbers, we have

$$b^{pq} \;=\; (b^p)^q \tag{6.1-6c}$$

This rule is a consequence of the first. And we shall extend it too into the realm of exponents that are not counting numbers.

Start out with the obvious:

$$b^1 \;=\; b \tag{6.1-7}$$

Let n be any counting number. Then by extension of the new rule we have

$$(b^{\frac{1}{n}})^n \;=\; b^{\frac{n}{n}} \;=\; b^1 \;=\; b \tag{6.1-8}$$

This equation doesn't tell us what $b^{\frac{1}{n}}$ is, or even if it exists. But it does provide an avenue for investigation. Consider the function,

$$f(x) \;=\; x^n \tag{6.1-9}$$

In your algebra studies, they told you that whenever b is positive, there is always an x that solves this equation, $x^n = b$. They called that x, the nth root of b. But they never proved that such an x exists. The following two paragraphs are optional material that provides that proof using the calculus you have learned so far.

Does what we've learned about calculus so far provide any clues about the behavior of the function, $f(x) = x^n$?

Suppose that $b > 0$. We know that $f(x) = x^n$ is a continuous function (remember that n is a counting number here. Click here to review how we know that this function is continuous). We also know that $f(0) = 0$ and that $f(x)$ can be made as large as you want by choosing x large enough. And that includes making x large enough so that $f(x) > b$. Call the x large enough to do that, x_{high}. So

$$f(x_{\text{high}}) \;>\; b \tag{6.1-10}$$

That means we can bracket $f(x)$ both above and below b. That is, $0 < b < f(x_{\text{high}})$. According to *The Intermediate Value Theorem*, that means that there must be a solution, x, to

$$f(x) \;=\; x^n \;=\; b \tag{6.1-11}$$

And that solution lies somewhere on $0 < x < x_{\text{high}}$.

Not only that, but $f(x) = x^n$ increases everywhere that $x > 0$. We know that because

$$f'(x) \;=\; nx^{n-1} \tag{6.1-12}$$

which is also continuous and always positive whenever $x > 0$. So once $f(x)$ reaches the solution to $f(x) = b$, it can never come back. This is consequence of *Rolle's Theorem*.

And that means that only a single positive x can satisfy $x^n = b$. For each positive b, there is only one positive nth root of b.

end of optional material

Clearly if for $b > 0$ you define

$$b^{\frac{1}{n}} \;=\; \sqrt[n]{b} \tag{6.1-13}$$

then you satisfy equation 6.1-8. And if you restrict applying exponents only to positive numbers, then only the nth root of b will *ever* satisfy 6.1-8. So that is how we define $b^{\frac{1}{n}}$ when b is positive and n is a counting number.

Clearly if $b^{\frac{1}{n}}$ is a positive real number whenever n is a counting number, then if m is some other counting number, you can raise $b^{\frac{1}{n}}$ to the mth power. As a consequence of equation 6.1-6c, we have

$$(b^{\frac{1}{n}})^m \;=\; b^{\frac{m}{n}} \tag{6.1-14a}$$

You can also readily see, by grouping terms, that

$$(b^{\frac{1}{n}})^m \;=\; (b^{\frac{1}{2n}})^{2m} \tag{6.1-14b}$$

and in general

$$b^{\frac{m}{n}} \;=\; b^{\frac{pm}{pn}} \tag{6.1-14c}$$

for any counting number, p. This, of course, is how we expect rational exponents to behave, since it is always true with rational numbers that

$$\frac{m}{n} \;=\; \frac{pm}{pn}$$

You can make *any* positive rational number by dividing one counting number by another. That is how we defined the rationals. So now we know how to take any positive real and raise it to any positive rational power. By applying equation 6.1-4b, you can also extend the definition in order to find any positive real taken to a negative rational power.

That still leaves one problem. We know there are plenty of real numbers that aren't rationals. How do you take a positive real number to a real power when that power isn't a rational number?

There is only one answer, and that is to use limits.

Recall that our original goal was to have exponential functions be continuous. Recall also that every real number, r, is the limit of a Cauchy sequence of rationals. That means that for some sequence of rational numbers, $q_1, q_2, q_3, \ldots$, it is true that

$$r = \lim_{k \to \infty} q_k \tag{6.1-15}$$

If b is a positive real number and we would like the function, $f(x) = bx$, to be continuous, then *The Stepping Stone Theorem* requires that

$$b^r = \lim_{k \to \infty} b^{q_k} \tag{6.1-16}$$

On the right hand side of the equation, we have b raised only to rational powers, and we have already demonstrated how we can define them. This limit is how we can define raising a positive real number to any real exponent and make the resulting exponential function continuous. All we have to do is prove that the limit always exists, which goes hand in hand with proving that exponentials are continuous. Such a proof is long and tortuous and will probably not be on the exam. But it involves only concepts that we have covered so far. So I am including it as optional material for those who are interested. For an interesting (but optional)journey, click here.

Coached Exercise

Isaac Newton observed that a heated object left out to cool does so according to the following formula:

$$T(t) - T_a = A\,b^{-t}$$

where t is time, $T(t)$ is the object's temperature as a function of time, T_a is the ambient temperature of the object's surroundings, and A and b are both positive constants (with $b > 1$).

Let t be in minutes and $T(t)$ be in degrees C. A loaf of bread is pulled from the oven and allowed to cool. It's initial temperature when it is pulled from the oven is 100°C. After 5 minutes it has cooled to 75°C. After 10 minutes it has cooled to 60°C. Find the ambient temperature of the room, T_a, and the two constants, A and b that make this loaf fit the temperature equation above?

Step 1: What does the problem tell you about $T(t)$? It gives you three different temperatures at three different times. That is, it tells you that at $t = 0$, the temperature is 100°C. So

$$T(0) = 100$$

Likewise, it tells you that at $t = 5$ and at $t = 10$ you have

$$T(5) \;=\; 75 \qquad \text{and} \qquad T(10) \;=\; 60$$

respectively.

Step 2: What equations can you make from this information? You can plug each of the values above into the original equation. For $t = 0$ you get

$$100 - T_a \;=\; A\,b^0$$

Likewise with the other two you can form the equations:

$$75 - T_a \;=\; A\,b^{-5} \qquad \text{and} \qquad 60 - T_a \;=\; A\,b^{-10}$$

Step 3: How can you eliminate a variable from these equations? Start out with the easiest equation, which is the first one. You know that no matter what b is, $b^0 = 1$. So the first equation becomes simply:

$$100 - T_a \;=\; A$$

That means that in the other two equations, everywhere you see A, you can substitute $100 - Ta$:

$$75 - T_a \;=\; (100 - T_a)\,b^{-5} \qquad \text{and} \qquad 60 - T_a \;=\; (100 - T_a)\,b^{-10}$$

Step 4: How can you eliminate yet another variable from the above two equations? There are several approaches at this point, but I will show you what I think is the easiest. If you divide both equations through by $(100 - T_a)$, you have

$$\frac{75 - T_a}{100 - T_a} \;=\; b^{-5} \qquad \text{and} \qquad \frac{60 - T_a}{100 - T_a} \;=\; b^{-10}$$

But observe that $b^{-10} = (b^{-5})^2$. Consequently if you square the first of the above two equations, you come up with an expression you can substitute for $b{-}10$ in the second:

$$\frac{60 - T_a}{100 - T_a} \;=\; \frac{(75 - T_a)^2}{(100 - T_a)^2}$$

or equivalently

$$(60 - T_a)\,(100 - T_a) \;=\; (75 - T_a)^2$$

Step 5: Multiply it out and solve for T_a.

$$6000 - 160\,T_a + {T_a}^2 \;=\; 5625 - 150\,T_a + {T_a}^2$$

The ${T_a}^2$'s cancel, and you are left with

$$375 \;=\; 10\,T_a \qquad \text{hence} \qquad 37.5 \;=\; T_a$$

Step 6: Back Substitute to find b^{-5}. You now know what T_a is, so you can plug it into other equations.

$$\frac{75 - 37.5}{100 - 37.5} \;=\; \frac{37.5}{62.5} \;=\; \frac{3}{5} \;=\; b^{-5}$$

So b is the 5th root of $\frac{5}{3}$, which is 1.107566343. And you know that $A = 100 - T_a$, so $A = 62.5$.

I leave it up to you to plug these numbers back into the original problem to see that they do indeed work.

Exercise

How much have you learned about the behavior of exponentials? Using methods similar to those in the text, can you prove that

$$a^r \times b^r \;=\; (ab)^r$$

when a and b are positive real numbers?

Start out by showing it is true for

$$a^n \times b^n \;=\; (ab)^n$$

when n is a counting number. Do this by making an argument about the grouping of terms (or if you feel ambitious, you can prove it by induction).

Next use a little algebra to show that it is true for

$$a^{-n} \times b^{-n} \;=\; (ab)^{-n}$$

as well by using the definition given in eq. 6.1-4b.

From there demonstrate that

$$a^{\frac{1}{n}} \times b^{\frac{1}{n}} \;=\; (ab)^{\frac{1}{n}}$$

This is a little more difficult, but see where you can get to starting with

$$(a^{\frac{1}{n}})^n \times (b^{\frac{1}{n}})^n \;=\; (a^{\frac{1}{n}}b^{\frac{1}{n}})^n$$

which is a consequence of the first part of this exercise. Then apply what we know about the product of exponents.

Once you have that, it should be easy to show that

$$a^{\frac{m}{n}} \times b^{\frac{m}{n}} \;=\; (ab)^{\frac{m}{n}}$$

whenever m and n are counting numbers. Use the results you have so far to show that.

Finally, use the continuity of exponentials (and the product rule for limits) to argue that

$$a^r \times b^r \;=\; (ab)^r$$

for any real number, r.